The Column and Row Hilbert Operator Spaces

Roy M. Araiza
Department of Mathematics
Purdue University

Abstract

Given a Hilbert space $\mathscr{H}$ we present the construction and some properties of the column and row Hilbert operator spaces $\mathscr{H}_c$ and $\mathscr{H}_r$, respectively. The main proof of the survey is showing that $\mathscr{C}\mathscr{B}(\mathscr{H}_c, \mathscr{K}_c)$ is completely isometric to $\mathscr{B}(\mathscr{H}, \mathscr{K})$, in other words, the completely bounded maps between the corresponding column Hilbert operator spaces is completely isometric to the bounded operators between the Hilbert spaces. A similar theorem for the row Hilbert operator spaces follows via duality of the operator spaces. In particular there exists a natural duality between the column and row Hilbert operator spaces which we also show. The presentation and proofs are due to Effros and Ruan [1].

1 The Column Hilbert Operator Space

Given a Hilbert space $\mathscr{H}$, we define the *column isometry* $C : \mathscr{H} \longrightarrow \mathscr{B}(\mathbb{C}, \mathscr{H})$ given by

$$\xi \mapsto C(\xi)\alpha := \alpha\xi, \alpha \in \mathbb{C}.$$

Then given any $n \in \mathbb{N}$, we define the nth-matrix norm on $M_n(\mathscr{H})$, by

$$\|\xi\|_{c,n} = \left\|C^{(n)}(\xi)\right\|, \xi \in M_n(\mathscr{H}).$$

These are indeed operator space matrix norms by Ruan's Representation theorem and thus we define the *column Hilbert operator space* as

$$\mathscr{H}_c = \left(\mathscr{H}, \left\{\|\cdot\|_{c,n}\right\}_{n\in\mathbb{N}}\right).$$

It follows that given $\xi \in M_n(\mathscr{H})$ we have that the adjoint operator of $C(\xi)$ is given by

$$C(\xi)^* : \mathscr{H} \longrightarrow \mathbb{C}, \zeta \mapsto (\zeta|\,\xi).$$

Suppose $C(\xi)^*(\zeta) = d, c \in \mathbb{C}$. Then we see that

$$c\overline{d} = (c|\,C(\xi)^*(\zeta)) = (c\xi|\,\zeta) = c\overline{(\zeta|\,\xi)} = (c|\,(\zeta|\,\xi)).$$

We are also able to compute the matrix norms on rectangular matrices. First note that for $\xi, \eta \in \mathscr{H}$,

$$C(\eta)^*C(\xi)\alpha = (\alpha\xi|\,\eta) = \alpha\,(\xi|\,\eta),$$

thus giving us the operator $C(\eta)^*C(\xi) := (\xi|\,\eta)$.

Now, for $m, n \in \mathbb{N}, \xi \in M_{m,n}(\mathscr{H}_c)$, we have the induced amplification

$$C^{(m,n)} : M_{m,n}(\mathscr{H}_c) \longrightarrow M_{m,n}(\mathscr{B}(\mathbb{C}, \mathscr{H})) \cong \mathscr{B}(\mathbb{C}^n, \mathscr{H}^m),$$

and thus, we have

$$\left\|C^{(m,n)}(\xi)\right\| = \left\|C^{(m,n)}(\xi)^*C^{(m,n)}(\xi)\right\|^{\frac{1}{2}} = \left\|[C(\xi_{ij})]^*\,[C(\xi_{ij})]\right\|^{\frac{1}{2}}$$

$$= \left\| \left[\sum_{\ell=1}^{n} C(\xi_{\ell i})^* C(\xi_{\ell j}) \right] \right\|^{\frac{1}{2}} = \left\| \left[\sum_{\ell=1}^{n} (\xi_{\ell j} | \, \xi_{\ell i}) \right] \right\|^{\frac{1}{2}}.$$

Note that we have used the isometry

$$\varphi : M_{m,n}(\mathscr{H}_c) \cong \mathscr{B}(\mathbb{C}^n, \mathscr{H}^m),$$

and since both structures are indeed operator spaces, we actually have that φ is a complete isometry of these operator space structures. This is the case since for all $p \in \mathbb{N}$,

$$M_p(M_{m,n}(\mathscr{H}_c)) = M_{pm,pn}(\mathscr{H}_c) \cong_{\varphi} \mathscr{B}(\mathbb{C}^{pn}, \mathscr{H}^{pm}) = M_p(\mathscr{B}(\mathbb{C}^n, \mathscr{H}^m)) \cong_{\varphi^{(p)}} M_p(M_{m,n}(\mathscr{H}_c)).$$

In particular we have

$$M_{m,1}(\mathscr{H}_c) \cong \mathscr{H}_c^m,$$

completely isometrically since

$$M_{m,1}(\mathscr{H}_c) \cong_C M_{m,1}(\mathscr{B}(\mathbb{C}, \mathscr{H})) \cong \mathscr{B}(\mathbb{C}, \mathscr{H}^m) \cong_C \mathscr{H}_c^m.$$

C denotes the column isometry on the corresponding spaces. We now wish to present another method for computing the matrix norms in the column Hilbert operator space. Suppose that $\alpha^{(h)} \in M_n$, with $(e_h)_{1 \le h \le p} \subset \mathscr{H}$ orthonormal vectors. We then have

$$\begin{aligned}
\left\| \sum_h \alpha^{(h)} \otimes e_h \right\|_c &= \left\| \left[\sum_h \alpha_{ij}^{(h)} e_h \right] \right\|_c \\
&= \left\| C^{(n)} \left(\left[\sum_h \alpha_{ij}^{(h)} e_h \right] \right) \right\| \\
&= \left\| \left[C \left(\sum_h \alpha_{ij}^{(h)} e_h \right) \right] \right\| \\
&= \left\| \left[C \left(\sum_h \alpha_{ij}^{(h)} e_h \right) \right]^* \left[C \left(\sum_h \alpha_{ij}^{(h)} e_h \right) \right] \right\|^{\frac{1}{2}} \\
&= \left\| \left[\sum_\ell \left(\sum_h \alpha_{\ell j}^{(h)} e_h \middle| \sum_g \alpha_{\ell i}^{(g)} e_g \right) \right] \right\|^{\frac{1}{2}} \\
&= \left\| \left[\sum_{h,\ell} \alpha_{\ell j}^{(h)} \overline{\alpha}_{\ell i}^{(h)} \right] \right\|^{\frac{1}{2}} \\
&= \left\| \sum_h \alpha^{(h)*} \alpha^{(h)} \right\|^{\frac{1}{2}} \\
&= \left\| \begin{bmatrix} \alpha^{(1)} \\ \vdots \\ \alpha^{(p)} \end{bmatrix} \right\|.
\end{aligned}$$

The same formula will also work for rectangular matrices.

2 The Row Hilbert Operator Space

We now wish to present the row Hilbert operator space. Recall that given a Hilbert space $\mathscr{H}$ we have the canonical isometry between the conjugate Hilbert space and its Banach dual given by

$$\theta : \overline{\mathscr{H}} \longrightarrow \mathscr{H}^*, \overline{\xi} \mapsto f_\xi, f_\xi(\zeta) := (\zeta | \, \xi) \, ,$$

and by the Riesz representation theorem we know that $\|f_\xi\| = \|\xi\|$. Define the *row isometry* by

$$R : \mathscr{H} \longrightarrow \mathscr{H}^{**} = \mathscr{B}(\mathscr{H}^*, \mathbb{C}) = \mathscr{B}(\overline{\mathscr{H}}, \mathbb{C}), \zeta \mapsto R(\zeta)(\overline{\xi}) = \theta(\overline{\xi})(\zeta) = (\zeta | \, \xi) \, .$$

Since $\mathscr{B}(\overline{\mathscr{H}}, \mathbb{C})$ is an operator space, then we have an induced operator space structure on $\mathscr{H}$. We define the *row Hilbert operator space* as

$$\mathscr{H}_r := \left(\mathscr{H}, \left\{ \|\cdot\|_{r,n} \right\}_n \right),$$

where if given $\xi \in M_n(\mathscr{H})$, then

$$\|\xi\|_r = \left\| R^{(n)}(\xi) \right\|, R^{(n)} : M_n(\mathscr{H}) \longrightarrow M_n(\mathscr{B}(\overline{\mathscr{H}}, \mathbb{C})) \cong \mathscr{B}(\overline{\mathscr{H}}^n, \mathbb{C}^n),$$

is the induced amplification. The adjoint operator of the row isometry is given by

$$R(\xi)^* : \mathbb{C} \longrightarrow \overline{\mathscr{H}}, a \mapsto a\overline{\xi}.$$

This is true since given $\zeta \in \mathscr{H}$,

$$\left(R(\xi)^* c | \, \overline{\zeta}\right) = c\overline{R(\xi)\overline{\overline{\zeta}}} = c\overline{\theta(\overline{\zeta})\xi} = c\,\overline{(\xi | \, \zeta)} = c \, (\zeta | \, \xi) = \left(c\overline{\xi} \middle| \, \overline{\zeta}\right) .$$

Thus, we have that for $\eta, \xi \in \mathscr{H}, c \in \mathbb{C}$, then $R(\xi) : \overline{\mathscr{H}} \longrightarrow \mathscr{C}, R(\eta)^* : \mathbb{C} \longrightarrow \overline{\mathscr{H}}$, and

$$R(\xi)R(\eta)^* c = R(\xi)(c\overline{\eta}) = c \, (\xi | \, \eta) \, ,$$

and thus $R(\xi)R(\eta)^* := (\xi | \, \eta)$.

As we did with the column isometry, let us compute the rectangular matrix norm for the row Hilbert operator space. Given $\xi \in M_{m,n}(\mathscr{H}_r)$, we then have

$$\|\xi\|_r = \left\| R^{(m,n)}(\xi) \right\| = \left\| [R(\xi_{ij})] \, [R(\xi_{ij})]^* \right\|^{\frac{1}{2}} = \left\| \left[\sum_{\ell=1}^m R(\xi_{i\ell}) R(\xi_{j\ell})^* \right] \right\|^{\frac{1}{2}} = \left\| \left[\sum_{\ell=1}^m (\xi_{i\ell} | \, \xi_{j\ell}) \right] \right\|^{\frac{1}{2}} .$$

We also see that

$$M_{1,n}(\mathscr{H}_r) \cong M_{1,n}(\mathscr{B}(\overline{\mathscr{H}}, \mathbb{C})) \cong \mathscr{B}(\overline{\mathscr{H}}^n, \mathbb{C}) \cong \mathscr{H}_r^n.$$

As for the column Hilbert operator space, we have the very convenient method for computing the matrix norm. Letting $\alpha^{(h)} \in M_m$, and $(e_h)_{1\le h\le p} \subset \mathscr{H}$ orthonormal vectors, we have

$$\begin{aligned}
\left\|\sum_h \alpha^{(h)} \otimes e_h\right\|_r &= \left\|\left[\sum_h \alpha_{ij}^{(h)} e_h\right]\right\|_r \\
&= \left\|R^{(m)}\left(\left[\sum_h \alpha_{ij}^{(h)} e_h\right]\right)\right\| \\
&= \left\|\left[R\left(\sum_h \alpha_{ij}^{(h)} e_h\right)\right]\right\| \\
&= \left\|\left[R\left(\sum_h \alpha_{ij}^{(h)} e_h\right)\right]\left[R\left(\sum_h \alpha_{ij}^{(h)} e_h\right)\right]^*\right\|^{\frac{1}{2}} \\
&= \left\|\left[\sum_\ell \left(\sum_h \alpha_{i\ell}^{(h)} e_h \middle| \sum_g \alpha_{j\ell}^{(g)} e_g\right)\right]\right\|^{\frac{1}{2}} \\
&= \left\|\left[\sum_{\ell,h} \alpha_{i\ell}^{(h)} \overline{\alpha}_{j\ell}^{(h)}\right]\right\|^{\frac{1}{2}} \\
&= \left\|\sum_h \alpha^{(h)} \alpha^{(h)*}\right\|^{\frac{1}{2}} \\
&= \left\|\left[\alpha^{(1)} \cdots \alpha^{(p)}\right]\right\|.
\end{aligned}$$

We point out the dualities between the column and row isometries;

$$\begin{aligned}
C &: \mathscr{H} \longrightarrow \mathscr{B}(\mathbb{C}, \mathscr{H}), \xi \mapsto C(\xi)\alpha = \alpha\xi, \\
R &: \mathscr{H} \longrightarrow \mathscr{H}^{**} = \mathscr{B}(\overline{\mathscr{H}}, \mathbb{C}), \xi \mapsto R(\xi)(\overline{\eta}) = (\xi|\, \eta)\,.
\end{aligned}$$

The corresponding adjoint mappings were defined for $\xi \in \mathscr{H}$ as

$$\begin{aligned}
C(\xi)^* &: \mathscr{H} \longrightarrow \mathbb{C}, \zeta \mapsto (\zeta|\, \xi)\,, \\
R(\xi)^* &: \mathbb{C} \longrightarrow \overline{\mathscr{H}}, \alpha \mapsto \alpha\overline{\xi}.
\end{aligned}$$

This then gives us

$$\begin{aligned}
C(\eta)^* C(\xi) &:= (\xi|\, \eta) \\
R(\xi) R(\eta)^* &:= (\xi|\, \eta)\,.
\end{aligned}$$

Suppose now that we once again have an orthonormal set of vectors $(e_h)_{1\le h\le p} \subset \mathscr{H}$. We will calculate the column and row matrix norms on the row matrix

$$\|[e_1 \cdots e_p]\| \in M_{1,n}(\mathscr{H}).$$

Using our calculations as before, we know that

$$\|[e_1 \cdots e_p]\|_c = \left\|\sum_{j=1}^p E_{1j} \otimes e_j\right\|_c = \left\|\sum_{j=1}^p E_{j1} E_{1j}\right\|^{\frac{1}{2}} = \|I\|^{\frac{1}{2}} = 1.$$

In contrast we see that

$$\|[e_1 \cdots e_p]\|_r = \left\| \sum_{j=1}^{p} E_{1j} \otimes e_j \right\|_r = \left\| \sum_{j=1}^{p} E_{1j} E_{j1} \right\|^{\frac{1}{2}} = \sqrt{p}.$$

Thus, we do have that the induced operator space structures on the Hilbert space $\mathscr{H}$ by $\mathscr{B}(\mathbb{C}, \mathscr{H})$, and $\mathscr{B}(\overline{\mathscr{H}}, \mathbb{C})$ are indeed different. Suppose that $\dim \mathscr{H} = 1$, thus $\mathscr{H} = \mathbb{C}$. We then have that given $\xi \in \mathbb{C}$, that

$$C(\xi)\zeta = \zeta\xi \tag{1}$$

$$R(\xi)\overline{\zeta} = (\xi|\,\zeta) = \xi\overline{\zeta}. \tag{2}$$

In particular we see that both the column and row isometries are just the multiplication action of the vector ξ. This then tells us that for $\xi \in M_n$, that $\|\xi\|_c = \|\xi\|_r = \|[\xi_{ij}]\|$.

Proposition 2.1. *Given two Hilbert spaces $\mathscr{H}, \mathscr{K}$, then we have the completely isometric identification*

$$\mathscr{B}(\mathscr{H}, \mathscr{K}) \cong \mathscr{C}\mathscr{B}(\mathscr{H}_c, \mathscr{K}_c).$$

Proof. Let $T = [T_{kl}] \in M_n(\mathscr{B}(\mathscr{H}, \mathscr{K})) \cong \mathscr{B}(\mathscr{H}^n, \mathscr{K}^n)$, and with this we define the induced operator

$$\widetilde{T} \in M_n(\mathscr{C}\mathscr{B}(\mathscr{H}_c, \mathscr{K}_c)) \cong \mathscr{C}\mathscr{B}(\mathscr{H}_c, M_n(\mathscr{K}_c)),$$

defined by

$$\mathscr{H}_c \ni \xi \mapsto_{\widetilde{T}} [T_{kl}(\xi)] \in M_n(\mathscr{K}_c).$$

Now, given $\xi \in M_p(\mathscr{H}_c)$ we may assume that $\xi = \sum_{j=1}^{r} \alpha_j \otimes f_j$, where $(f_j)_{j=1}^{r} \subset \mathscr{H}$ are orthonormal vectors. Let $\mathscr{H}_o = \operatorname{span}_j f_j$ and let $\mathscr{K}_o = \operatorname{span}_{k,l} T_{kl}(\mathscr{H}_o)$ and let $(g_i)_{i=1}^{q} \subset \mathscr{K}_o$ be an orthonormal basis for $\mathscr{K}_o$. We define the following complex numbers $T_{kl}(i,j)$ as the coefficients given by

$$T_{kl}(f_j) = \sum_i T_{kl}(i,j) g_i,$$

and we set $T_o(i,j) = [T_{kl}(i,j)]_{k,l} \in M_n, T_o = [T_o(i,j)]_{i,j} \in M_{nq,nr}$. We see that $\|T_o\| \leq \|T\|$ since T_o is merely the restriction of T to $\mathscr{H}_o^n$. Define the following pth-amplification

$$\widetilde{T}^{(p)} = \mathrm{Id}_{M_p} \otimes \widetilde{T} : M_p \otimes \mathscr{H}_c \longrightarrow M_p \otimes M_n \otimes \mathscr{K}_c.$$

Then we compute

$$\widetilde{T}^{(p)}(\xi) = \sum_j \alpha_j \otimes \widetilde{T}(f_j) = \sum_j \alpha_j \otimes \sum_{k,l} E_{kl} \otimes T_{kl}(f_j)$$

$$= \sum_{i,j,k,l} \alpha_j \otimes E_{kl} \otimes T_{kl}(i,j) g_i = \sum_{i,j,k,l} \alpha_j \otimes T_{kl}(i,j) E_{kl} \otimes g_i = \sum_{i,j} \alpha_j \otimes T_o(i,j) \otimes g_i \in M_p \otimes M_n \otimes \mathscr{K}_c.$$

Thus, in computing the norm we see

$$\left\|\widetilde{T}^{(p)}(\xi)\right\|_{M_{pn}(\mathscr{K}_c)} = \left\| \begin{bmatrix} \sum_j \alpha_j \otimes T_o(1,j) \\ \vdots \\ \sum_j \alpha_j \otimes T_o(q,j) \end{bmatrix} \right\| = \left\| \begin{bmatrix} \mathrm{Id}_{M_p} \otimes T_o(1,1) & \cdots & \mathrm{Id}_{M_p} \otimes T_o(1,r) \\ \vdots & & \vdots \\ \mathrm{Id}_{M_p} \otimes T_o(q,1) & \cdots & \mathrm{Id}_{M_p} \otimes T_o(q,r) \end{bmatrix} \begin{bmatrix} \alpha_1 \otimes \mathrm{Id}_{M_n} \\ \vdots \\ \alpha_r \otimes \mathrm{Id}_{M_n} \end{bmatrix} \right\|$$

$$\leq \left\|\mathrm{Id}_{M_p} \otimes T_o\right\| \left\| \begin{bmatrix} \alpha_1 \otimes \mathrm{Id}_{M_n} \\ \vdots \\ \alpha_r \otimes \mathrm{Id}_{M_n} \end{bmatrix} \right\| \leq \|T\| \, \|\xi\|_{M_p(\mathscr{H}_c)}.$$

We then have $\left\|\widetilde{T}\right\|_{cb} \leq \|T\|$.

We need only show now that $\|T\| \leq \left\|\widetilde{T}\right\|_{cb}$. Begin by taking $\xi = (\xi_l)_l \in \mathscr{H}^n$. Recall that we have the identification

$$M_{n,1}(\mathscr{H}_c) \cong M_{n,1}(\mathscr{B}(\mathbb{C}, \mathscr{H})) \cong \mathscr{B}(\mathbb{C}, \mathscr{H}^n) \cong \mathscr{H}_c^n.$$

Let $(e_j)_{j=1}^p \subset \mathscr{H}$ be an orthonormal basis for $\operatorname{span}_l \xi_l$, and therefore we know that we may write for each $l, 1 \leq l \leq n$,

$$\xi_\ell = \sum_{j=1}^{p} (\xi|\, e_j)\, e_j, \sum |\, (\xi|\, e_j)|^2 = \|\xi\|^2 .$$

Now, $T = [T_{kl}] \in M_n(\mathscr{B}(\mathscr{H}, \mathscr{K}))$, and thus we see that

$$T(\xi) = \left[\sum_{l,j} T_{kl}(e_j)\ (\xi_l|\, e_j)\right] \in M_{n,1}(\mathscr{K}_c).$$

Thus, at this point we decompose the matrix as

$$\left[\sum_{l,j} T_{kl}(e_j)\ (\xi_l|\, e_j)\right] = \left[[T_{kl}(e_1)]_{kl} \cdots [T_{kl}(e_p)]_{kl}\right] \begin{bmatrix} (\xi_1|\, e_1) \\ \vdots \\ (\xi_l|\, e_j) \\ \vdots \\ (\xi_n|\, e_p) \end{bmatrix},$$

where

$$\left[[T_{kl}(e_1)]_{kl} \cdots [T_{kl}(e_p)]_{kl}\right] \in M_{n,np}(\mathscr{K}_c), \begin{bmatrix} (\xi_1|\, e_1) \\ \vdots \\ (\xi_l|\, e_j) \\ \vdots \\ (\xi_n|\, e_p) \end{bmatrix} \in M_{np,1}.$$

We have the induced map $\widetilde{T} : \mathscr{H}_c \longrightarrow M_n(\mathscr{K}_c), \zeta \mapsto [T_{kl}(\zeta)]$, and therefore the first matrix becomes

$$\widetilde{T}^{(1,p)}([e_1 \cdots e_p]),$$

and if we write the coefficient matrix as

$$[\,(\xi_l|\, e_j)]_{l,j}\,,$$

then we have the following inequalities;

$$\|T(\xi)\| \leq \left\|\widetilde{T}^{(1,p)}([e_1 \cdots e_p])\right\| \|[\,(\xi_l|\, e_j)]\| \leq \left\|\widetilde{T}\right\|_{cb} \|[e_1 \cdots e_p]\|_c \|\xi\| = \left\|\widetilde{T}\right\|_{cb} \|\xi\| .$$

Thus, we have the desired equality and have shown that bounded linear operators between two Hilbert spaces are completely isometric to the completely bounded maps between the corresponding column Hilbert operator spaces. □

We now have the following dualities;

$$(\mathscr{H}_c)^* \cong \mathscr{C}\mathscr{B}(\mathscr{H}_c, \mathbb{C}) \cong \mathscr{B}(\mathscr{H}, \mathbb{C}) \cong \mathscr{B}(\mathscr{H}^{**}, \mathbb{C}) \cong \mathscr{B}(\overline{\mathscr{H}^*}, \mathbb{C}) \cong (\mathscr{H}^*)_r.$$

Letting $\mathscr{K} = \mathscr{H}^*$, we then have

$$(\mathscr{K}_r)^* \cong \mathscr{H}_c^{**} \cong \mathscr{H}_c \cong (\mathscr{K}^*)_c,$$

where these dualites are complete isometries.

Proposition 2.2. *Given two Hilbert spaces $\mathscr{H}, \mathscr{K}$ we have the following completely isometric identification;*

$$\mathscr{B}(\mathscr{K}^*, \mathscr{H}^*) \cong \mathscr{C}\mathscr{B}(\mathscr{H}_r, \mathscr{K}_r).$$

By our dualites already stated, we then have the completely isometric identifications

$$(\mathscr{H}_c)^* \cong (\mathscr{H}^*)_r \cong (\overline{\mathscr{H}})_r.$$

Proposition 2.3. *Given two Hilbert spaces $\mathscr{H}, \mathscr{K}$, we have the completely isometric identification*

$$\overline{\mathscr{H}_c} \cong (\overline{\mathscr{H}})_c, \overline{\mathscr{H}_r} \cong (\overline{\mathscr{H}})_r.$$

References

[1] Edward G Effros and Zhong-Jin Ruan. Operator spaces. 2000.